Oscilador armónico II

$H = \frac{P^2}{2m} + \frac{1}{2} m\omega^2 X^2$

- Definimos a (descenso; ojo) y a^+ (ascenso); $[a, a^+] = 1$, $[N, a] = -a$, $[N, a^+] = a^+$

- $N = a^+ a \neq a a^+$

N es hermitiano; $H = \hbar\omega(N + \frac{1}{2})$

$N|\phi_n\rangle = n|\phi_n\rangle$ $n \geq 0$ entero

$a|\phi_0\rangle = 0$
$a|\phi_n\rangle$ e-vector de N con e-valor $n-1$
$a^+|\phi_n\rangle$ e-vector de N con e-valor $n+1$

$$a = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\hat{X} + i\frac{1}{\sqrt{m\omega\hbar}}\hat{P}\right)$$

$$a^+ = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\hat{X} - i\frac{1}{\sqrt{m\omega\hbar}}\hat{P}\right)$$

5.6. Some Theorems

Theorem 15. There is no degeneracy in one-dimensional bound states.

11.2. Eigenvectores de H

- El vector $|\phi_0\rangle$ es aquel que satisface $a|\phi_0\rangle = 0$. Esto determina a $|\phi_0\rangle$ salvo por una ~~constante que fijamos al normalizarlo.~~ escalar complejo cuya magnitud fijamos al normalizarlo.
- El vector $|\phi_1\rangle$ que corresponde a $n = 1$ cumple

$$|\phi_1\rangle = c_1 a^\dagger |\phi_0\rangle$$

Podemos determinar c_1 al pedir que $|\phi_1\rangle$ esté normalizado.

$$\langle\phi_1|\phi_1\rangle = |c_1|^2 \langle\phi_0|aa^\dagger|\phi_0\rangle = |c_1|^2 \langle\phi_0|a^\dagger a + 1|\phi_0\rangle = |c_1|^2 = 1$$

- Podemos elegir a la fase de c_1 real y así tenemos $c_1 = 1$
- Así $|\phi_1\rangle = a^\dagger |\phi_0\rangle$.
- Similarmente podemos construir a $|\phi_2\rangle$ a partir de $|\phi_1\rangle$.

$$|\phi_2\rangle = c_2 a^\dagger |\phi_1\rangle$$

- Al pedir que $|\phi_2\rangle$ esté normalizado y elegir la fase de c_2 como real y positiva tenemos que

$$\begin{aligned}\langle\phi_2|\phi_2\rangle &= |c_2|^2 \langle\phi_1|aa^\dagger|\phi_1\rangle \\ &= |c_2|^2 \langle\phi_1|a^\dagger a + 1|\phi_1\rangle \\ &= 2|c_2|^2 = 1\end{aligned}$$

Entonces

$$\langle\phi_2|\phi_2\rangle = \frac{1}{\sqrt{2}} a^\dagger |\phi_1\rangle = \frac{1}{\sqrt{2}} (a^\dagger)^2 |\phi_0\rangle$$

- Podemos generalizar este proceso para construir a todos los e.V. Si conocemos a $|\phi_{n-1}\rangle$ (normalizado) entonces

$$|\phi_n\rangle = c_n a^\dagger |\phi_{n-1}\rangle$$

Al normalizar

$$\langle\phi_n|\phi_n\rangle = |c_n|^n \langle\phi_n|aa^\dagger|\phi_n\rangle = \cdots = n|c_n|^2 = 1$$

por lo que $c_n = 1/\sqrt{n}$

- Así, podemos obtener $|\phi_n\rangle$ a partir de $|\phi_0\rangle$

$$\begin{aligned}|\phi_n\rangle &= \frac{1}{\sqrt{n}} a^\dagger |\phi_{n-1}\rangle = \frac{1}{\sqrt{n}} \frac{1}{\sqrt{n-1}} (a^\dagger)^2 |\phi_{n-2}\rangle = \ldots \\ &= \frac{1}{\sqrt{n}} \frac{1}{\sqrt{n-1}} \cdots \frac{1}{\sqrt{2}} (a^\dagger)^n |\phi_0\rangle \qquad = \frac{1}{\sqrt{n!}} \left(a^\dagger\right)^n |\phi_0\rangle\end{aligned}$$

- Como H es hermitiano y sus e.v. son distintos entonces todos los kets $|\phi_n\rangle$ que corresponden a distintos valores de n son ortonormales.

$$\langle\phi_{n'}|\phi_n\rangle = \delta_{nn'} \quad \Rightarrow \quad \sum |\varphi_n\rangle\langle\varphi_n| = \mathbb{1}$$

- Podemos probar esta usando $|\phi_n\rangle = \frac{1}{\sqrt{n!}}\left(a^\dagger\right)^n|\phi_0\rangle$. pero no es necesario.
- Los observables X y P son combinación de a y $a^\dagger$ así que las cantidades de interés se pueden expresar en términos de estos.
- Es importante encontrar el efecto de a y $a^\dagger$ sobre $|\phi_n\rangle$

$n \to n+1$

$$|\phi_n\rangle = \frac{1}{\sqrt{n}} a^+ |\phi_{n-1}\rangle \Rightarrow a^+|\phi_n\rangle = \sqrt{n+1}\,|\phi_{n+1}\rangle$$

mult por a

$$|\phi_n\rangle = \frac{1}{\sqrt{n}} a^+ |\phi_{n-1}\rangle \Rightarrow a|\phi_n\rangle = \frac{1}{\sqrt{n}} a a^+ |\phi_{n-1}\rangle$$

$$= \frac{1}{\sqrt{n}}(a^+a+1)|\phi_{n-1}\rangle$$

$$= \sqrt{n}\,|\phi_{n-1}\rangle$$

$$\boxed{\begin{aligned} a^+|\phi_n\rangle &= \sqrt{n+1}\,|\phi_{n+1}\rangle \\ a|\phi_n\rangle &= \sqrt{n}\,|\phi_{n-1}\rangle \end{aligned}}$$

The adjoint equations of (C-19a) and (C-19b) are:

$$\langle\varphi_n|a = \sqrt{n+1}\,\langle\varphi_{n+1}| \tag{C-21a}$$

$$\langle\varphi_n|a^\dagger = \sqrt{n}\,\langle\varphi_{n-1}| \tag{C-21b}$$

Note that a decreases or increases n by one unit depending on whether it acts on the ket $|\varphi_n\rangle$ or on the bra $\langle\varphi_n|$. Similarly, $a^\dagger$ increases or decreases n by one unit, depending on whether it acts on the ket $|\varphi_n\rangle$ or on the bra $\langle\varphi_n|$.

$$X|\varphi_n\rangle = \sqrt{\frac{\hbar}{m\omega}}\frac{1}{\sqrt{2}}(a^\dagger + a)|\varphi_n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n+1}\,|\varphi_{n+1}\rangle + \sqrt{n}\,|\varphi_{n-1}\rangle\right] \tag{C-22a}$$

$$P|\varphi_n\rangle = \sqrt{m\hbar\omega}\frac{i}{\sqrt{2}}(a^\dagger - a)|\varphi_n\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\left[\sqrt{n+1}\,|\varphi_{n+1}\rangle - \sqrt{n}\,|\varphi_{n-1}\rangle\right] \tag{C-22b}$$

$$\langle\varphi_{n'}|a|\varphi_n\rangle = \sqrt{n}\,\delta_{n',n-1} \tag{C-23a}$$

$$\langle\varphi_{n'}|a^\dagger|\varphi_n\rangle = \sqrt{n+1}\,\delta_{n',n+1} \tag{C-23b}$$

$$\langle\varphi_{n'}|X|\varphi_n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n+1}\,\delta_{n',n+1} + \sqrt{n}\,\delta_{n',n-1}\right] \tag{C-23c}$$

$$\langle\varphi_{n'}|P|\varphi_n\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\left[\sqrt{n+1}\,\delta_{n',n+1} - \sqrt{n}\,\delta_{n',n-1}\right] \tag{C-23d}$$

$$(a) = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & .. & 0 & .. \\ 0 & 0 & \sqrt{2} & 0 & .. & 0 & .. \\ 0 & 0 & 0 & \sqrt{3} & .. & 0 & .. \\ 0 & 0 & 0 & 0 & .. & 0 & .. \\ .. & .. & .. & .. & .. & \sqrt{n} & .. \\ 0 & 0 & 0 & 0 & 0 & 0 & .. \\ .. & .. & .. & .. & .. & .. & .. \end{pmatrix} \qquad (a^\dagger) = \begin{pmatrix} 0 & 0 & 0 & 0 & .. & 0 & .. \\ \sqrt{1} & 0 & 0 & 0 & .. & 0 & .. \\ 0 & \sqrt{2} & 0 & 0 & .. & 0 & .. \\ 0 & 0 & \sqrt{3} & 0 & .. & 0 & .. \\ .. & .. & .. & .. & .. & 0 & .. \\ 0 & 0 & 0 & 0 & \sqrt{n+1} & 0 & .. \\ .. & .. & .. & .. & .. & .. & .. \end{pmatrix}$$

$$a = \sum_{n=1}^{\infty} \sqrt{n}\,|n-1\rangle\langle n| \qquad a^+ = \sum_{n=0}^{\infty} \sqrt{n+1}\,|n+1\rangle\langle n|$$

Funciones de onda

- Usando la definición de a obtenemos

$$\frac{1}{\sqrt{2}}\left[\sqrt{\frac{m\omega}{\hbar}}X + \frac{i}{\sqrt{m\hbar\omega}}P\right]|\phi_0\rangle = 0$$

que en la base $\{|x\rangle\}$ resulta en

$$\left(\frac{m\omega}{\hbar}x + \frac{\mathrm{d}}{\mathrm{d}x}\right)\phi_0(x), \quad (*)$$

con $\phi_0(x) = \langle x|\phi_0\rangle$.

- La solución a esta ecuación diferencial de primer orden es

$$\phi_0(x) = ce^{-\frac{m\omega}{2\hbar}x^2}$$

de $(*)$.

normalización

$$1 = \int |\phi_0(x)|^2 dx$$
$$= \int |c|^2 e^{-\frac{m\omega}{\hbar}x^2} dx$$
$$\sqrt{\frac{m\omega}{\hbar}}\, x \to s$$
$$= |c|^2 \sqrt{\frac{\hbar}{m\omega}} \int e^{-s^2} ds = \sqrt{\frac{\hbar\pi}{m\omega}}|c|^2$$

$\sqrt{\pi}$

$$\Rightarrow \quad c = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4}$$

Usando $\quad |\phi_n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|\phi_0\rangle$

$$\varphi_n(x) = \langle x|\varphi_n\rangle = \frac{1}{\sqrt{n!}}\langle x|(a^\dagger)^n|\varphi_0\rangle$$
$$= \frac{1}{\sqrt{n!}}\frac{1}{\sqrt{2^n}}\left[\sqrt{\frac{m\omega}{\hbar}}x - \sqrt{\frac{\hbar}{m\omega}}\frac{\mathrm{d}}{\mathrm{d}x}\right]^n \varphi_0(x)$$

that is:

$$\varphi_n(x) = \left[\frac{1}{2^n n!}\left(\frac{\hbar}{m\omega}\right)^n\right]^{1/2}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[\frac{m\omega}{\hbar}x - \frac{\mathrm{d}}{\mathrm{d}x}\right]^n \mathrm{e}^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2} \qquad \text{(C-27)}$$

It is easy to see from this expression that $\varphi_n(x)$ is the product of $\mathrm{e}^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2}$ and a polynomial of degree n and parity $(-1)^n$, called a *Hermite polynomial* (*cf.* Complements B_V and C_V).

A simple calculation gives the first several functions $\varphi_n(x)$:

$$\varphi_1(x) = \left[\frac{4}{\pi}\left(\frac{m\omega}{\hbar}\right)^3\right]^{1/4} x\,\mathrm{e}^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2}$$
$$\varphi_2(x) = \left(\frac{m\omega}{4\pi\hbar}\right)^{1/4}\left[2\frac{m\omega}{\hbar}x^2 - 1\right]\mathrm{e}^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2} \qquad \text{(C-28)}$$

n aumenta

- más ancha
- más ceros $\Rightarrow$ más energía

$$\frac{1}{2m}\langle p^2\rangle = -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty}\phi_n^*(x)\frac{d^2}{dx^2}\phi_n(x)\,dx$$

- Más densidad en los extremos (como en caso clásico)